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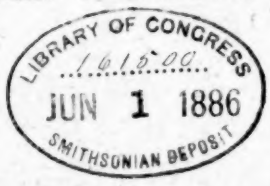

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THE ATTRACTION OF A RIGHT CIRCULAR CYLINDER ON A PARTICLE.

By CHAS. H. KUMMELL, Washington, D. C.

[CONTINUED FROM VOL. II, PAGE 38.]

One of the best illustrations of the advantage of this notation is the imaginary transformation of Jacobi, which may be given because it will be used in the reduction of (4).

Assume in (5) $\sin\varphi = i\tan\psi,$

or $$\cos\varphi = \frac{1}{\cos\psi},$$

or $$\Delta\varphi = \frac{\Delta(\psi_\beta)_{-\beta}}{\cos\psi}; \tag{7}$$

then $$u = \int_0^{\varphi}\frac{d\varphi}{\Delta\varphi} = i\int_0^{\psi}\frac{d\psi}{\Delta(\psi_\beta)_{-\beta}}, \tag{8}$$

or $$u = \varphi_\gamma = i\psi_\beta; \tag{9}$$

$$\therefore\ u_{-\gamma} = \varphi = (i\psi_\beta)_{-\gamma}; \tag{9'}$$

also $$\left(\frac{u}{i}\right)_{-\beta} = \left(\frac{\varphi_\gamma}{i}\right)_{-\beta} = \psi; \tag{9''}$$

therefore $$\sin u_{-\gamma} = i\tan\left(\frac{u}{i}\right)_{-\beta} = \frac{1}{i}\tan(iu)_{-\beta}.$$

$$\cos u_{-\gamma} = \frac{1}{\cos\left(\frac{u}{i}\right)_{-\beta}} = \frac{1}{\cos(iu)_{-\beta}},$$

$$\Delta u_{-\gamma} = \frac{\Delta\left(\frac{u}{i}\right)_{-\beta}}{\cos\left(\frac{u}{i}\right)_{-\beta}} = \frac{\Delta(iu)_{-\beta}}{\cos(iu)_{-\beta}}. \tag{7'}$$

Jacobi's type for the integral of the third species is with our notation:

$$\Pi(\varphi_\gamma, a_\gamma) = \int_0^\varphi \frac{\gamma^2 \sin a \cos a \, \Delta a \sin^2\varphi \, d\varphi}{(1-\gamma^2\sin^2 a \sin^2\varphi)\,\Delta\varphi}$$

$$= \frac{\Delta a}{\tan a}\left(\int_0^\varphi \frac{d\varphi}{(1-\gamma^2\sin^2 a\sin^2\varphi)\,\Delta\varphi} - \int_0^\varphi \frac{d\varphi}{\Delta\varphi}\right)$$

$$= \frac{\Delta a}{\tan a}[\Pi_1(\varphi, -\gamma^2\sin^2 a)^* - \varphi_\gamma]. \tag{10}$$

Comparing this with the third term of (4) we have $\varphi = \lrcorner, \gamma = \frac{c_e}{a_e}$, and

$$\sin^2 a = -\frac{e^2}{(d-r)^2}. \tag{11}$$

The angle a must be purely imaginary and if we assume by (7)

$$\sin a = i\tan\left(\frac{a_\gamma}{i}\right)_{-\beta} = i\tan \mathrm{A}, \text{ say, where A is real,} \tag{12}$$

then by (9') $$a = (i\mathrm{A}_\beta)_{-\gamma}. \tag{13}$$

We have by (11) and (12) $$\tan \mathrm{A} = \frac{e}{d-r}; \tag{14}$$

$$\therefore \sin \mathrm{A} = \frac{e}{b_e}, \quad \cos \mathrm{A} = \frac{d-r}{b_e}, \quad \Delta(\mathrm{A}_\beta)_{-\beta} = \frac{d+r}{a_e}. \tag{15}$$

We have then in (4), regarding (10),

$$A_e = m\vartheta\frac{eb_e}{d\sin^2\mathrm{A}}\Big\{\lrcorner_\gamma - \sin^2\mathrm{A}\,E(\lrcorner_\gamma) - \Delta(\mathrm{A}_\beta)^2_{-\beta}\Big[\lrcorner_\gamma + \frac{\tan(i\mathrm{A}_\beta)_{-\gamma}}{\Delta(i\mathrm{A}_\beta)_{-\gamma}}\,\Pi(\lrcorner_\gamma, i\mathrm{A}_\beta)\Big]\Big\}$$

$$= m\vartheta\frac{eb_e}{d\sin^2\mathrm{A}}\Big[[\gamma^2\lrcorner_\gamma - E(\lrcorner_\gamma)]\sin^2\mathrm{A} - i\sin\mathrm{A}\cos\mathrm{A}\,\Delta(\mathrm{A}_\beta)_{-\beta}\,\Pi(\lrcorner_\gamma, i\mathrm{A}_\beta)\Big]$$

*Legendre's notation is used.

$$= m\vartheta\frac{eb_e}{d}\left[\gamma^2 \Delta_\gamma - E(\Delta_\gamma) - i\frac{\Delta(A_\beta)_{-\beta}}{\tan A}\Pi(\Delta_\gamma, iA_\beta)\right]. \tag{16}$$

Jacobi's theorem for exchanging amplitude with parametric angle, which is so easily remembered by its symmetry, is

$$\Pi(\varphi_\gamma, a_\gamma) - \Pi(a_\gamma, \varphi_\gamma) = \varphi_\gamma E(a_\gamma) - a_\gamma E(\varphi_\gamma). \tag{17}$$

If in this we place $\varphi = \Delta$, then since

$$\Pi(a_\gamma, \Delta_\gamma) = \int_0^\varphi \frac{\gamma^2 \sin\Delta\cos\Delta\,\Delta\Delta\sin^2\varphi\,d\varphi}{(1-\gamma^2\sin^2 a\sin^2\varphi)\,\Delta\varphi} = 0,$$

we have
$$\Pi(\Delta_\gamma, a_\gamma) = \Delta_\gamma E(a_\gamma) - E(\Delta_\gamma), \tag{18}$$

and thus the quadrantal integral of the third species is in a very simple manner expressed in terms of integrals of the first and second species if a is real. Jacobi has shown that all possible cases of parameter are contained in his type. In the present case we have

$$\Pi(\Delta_\gamma, iA_\beta) = \Delta_\gamma E(iA_\beta) - iA_\beta E(\Delta_\gamma). \tag{19}$$

It is necessary to express $E(iA_\beta)$ in real terms.

We have $E(iA_\beta) = \int_0^{(iA_\beta)_{-\gamma}} d\varphi\,\Delta\varphi = i\int_0^A \frac{d\varphi}{\Delta(\varphi_\beta)_{-\beta}} \cdot \frac{\Delta(\varphi_\beta)^2_{-\beta}}{\cos^2\varphi}$ by (8) and (7′)

$$= i\tan A\,\Delta(A_\beta)_{-\beta} + i\int_0^A \tan\varphi \cdot \frac{\gamma^2\sin\varphi\cos\varphi\,d\varphi}{\Delta(\varphi_\beta)_{-\beta}}$$

$$= i\tan A\,\Delta(A_\beta)_{-\beta} + iA_\beta - iE(A_\beta). \tag{20}$$

With this we have in (19)

$$\Pi(\Delta_\gamma, iA_\beta) = i\left[\tan a\,\Delta(A_\beta)_{-\beta} + A_\beta - E(A_\beta)\right]\Delta_\gamma - iA_\beta E(\Delta_\beta);$$

and, using this in (16), we have

$$A_e = m\vartheta\frac{eb_e}{d}\times$$

$$\left\{\gamma^2\Delta_\gamma - E(\Delta_\gamma) + \Delta(A_\beta)^2_{-\beta}\Delta_\gamma - \frac{\Delta(A_\beta)_{-\beta}}{\tan A}\left[\Delta_\gamma A_\beta - A_\beta E(\Delta_\gamma) - \Delta_\gamma E(A_\beta)\right]\right\}$$

$$= m\vartheta\frac{eb_e}{d}\times \tag{21}$$

$$\left\{(1+\gamma^2\cos^2 A)\Delta_\gamma - E(\Delta_\gamma) - \frac{\Delta(A_\beta)_{-\beta}}{\tan A}\left[\Delta_\gamma A_\beta - A_\beta E(\Delta_\gamma) - \Delta_\gamma E(A_\beta)\right]\right\}.$$

This form is more lengthy than (4), but has the advantage of being expressed entirely in terms of tabulated functions. If the attracted particle is at midheight, then $e = h - e$ and the total attraction $A = 2A_e$. If e is infinitesimal, we obtain the attraction of a circular plate on a particle in its plane which may be expressed

$$A = 2\left(A_0 + \frac{dA_e}{de}\delta e\right), \tag{22}$$

where $2\delta e$ = thickness of the plate. But $A_0 = 0$; hence

$$A = 2\frac{dA_e}{de}\delta e \tag{22'}$$

$$= 2m\frac{\vartheta}{d}\delta e\left\{\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{\left(\frac{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}{(\rho^2+e^2)}\right)}\right.$$

$$\left. - e^2\int_{d-r}^{d+r}\frac{d\rho}{\rho}\sqrt{\left(\frac{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}{(\rho^2+e^2)^3}\right)}\right\}_{e=0}$$

$$= 2m\frac{\vartheta}{d}\delta e\int_{d-r}^{d+r}\frac{d\rho}{\rho^2}\sqrt{[(d+r)^2-\rho^2][\rho^2-(d-r)^2]}$$

$$\left(\rho = \frac{d-r}{\Delta\varphi};\ \Delta\varphi = \sqrt{\left(1 - \frac{4rd}{(d+r)^2}\sin^2\varphi\right)}\right.$$

$$= 2m\frac{\vartheta}{d}\delta e\,(d+r)\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\Delta\varphi^3}(1-\Delta\varphi^2)\left[\Delta\varphi^2 - \left(\frac{d-r}{d+r}\right)^2\right]$$

$$= 2m\frac{\vartheta}{d}\delta e\,(d+r)\left[-\left(\frac{d-r}{d+r}\right)^2\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\Delta\varphi^3} + 2\frac{d^2+r^2}{(d+r)^2}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\Delta\varphi} - \int_0^{\frac{\pi}{2}}d\varphi\,\Delta\varphi\right]$$

$$= 4m\frac{\vartheta}{d}\delta e\left(\frac{d^2+r^2}{d+r}F - (d+r)E\right)$$

$$\left(\gamma = \frac{2\sqrt{(rd)}}{d+r};\ \beta = \frac{d-r}{d+r}\right)$$

$$= 4m\frac{\vartheta}{d}\delta e\,(d+r)[(1+\beta^2)\Delta_\gamma - E(\Delta_\gamma)]. \tag{22''}$$

This form could, by evaluation of the indeterminate part, be derived from (21), which is to be doubled first.

If we follow Price's method of determining the attraction of plates (Price's *Infinitesimal Calculus*, Vol. III, p. 275) we obtain the identical result. But Price, in example of the attraction of a rectangular plate, commits a manifest blunder in the making the attraction on a particle at either one of its corners infinite. The corresponding case here is the attraction on a particle at the rim of the plate for which $d = r$; $\therefore \beta = 0$; $\gamma = 1$. Now, since

$$\int_0^{\lrcorner} \frac{d\varphi}{\lrcorner(\varphi_1)_{-1}} = \int_0^{\lrcorner} \frac{d\varphi}{\cos\varphi} = l \tan \tfrac{1}{2}(\lrcorner + \varphi)\Big]_0^{\lrcorner} = \infty,$$

we have also here apparently an infinite attraction, which is, however, multiplied by δe, which should be considered infinitesimal. This product, $\delta e \times \lrcorner_1$, could, however, not be evaluated because as it is introduced in the formula δe is perfectly arbitrary. The attraction of a circular plate of sensible thickness, $2\delta e$, on a particle at the circumference is evidently obtained by placing in (4) $d = r$; $\therefore a_e = 1\ (4r^2 + \delta e^2)$; $b_e = \delta e$; $c_e = 2r$; then

$$A = 2m\vartheta \frac{\delta e}{r} \sqrt{(4r^2 + \delta e^2)} \int_0^{\lrcorner} \frac{d\varphi}{\lrcorner\varphi^3} \left(\lrcorner\varphi^2 - \frac{\delta e^2}{4r^2 + \delta e^2} \right)$$

$$= 2m\vartheta \frac{\delta e}{r} \sqrt{(4r^2 + \delta e^2)} \left(\int_0^{\lrcorner} \frac{d\varphi}{\lrcorner\varphi} - \frac{\delta e^2}{4r^2 + \delta e^2} \int_0^{\lrcorner} \frac{d\varphi}{\lrcorner\varphi^3} \right)$$

$$\left(\gamma = \frac{2r}{1\ (4r^2 + \delta e^2)} \right) = 2m\vartheta \frac{\delta e}{r} \sqrt{(4r^2 + \delta e^2)} [\lrcorner_\gamma - E(\lrcorner_\gamma)]. \qquad (23)$$

As long as δe is finite, this value of the attraction of a cylinder is finite. From this it is evident already that the attraction of a plate, being necessarily less, *cannot be infinite*. But it may also be shown that the first term of (23), although appearing under an indeterminate form if $\delta e = 0$, is $= 0$. For we have

$$\left. \frac{\int_0^{\lrcorner} \frac{d\varphi}{\sqrt{\left(1 - \frac{4r^2}{4r^2 + \delta e^2} \sin^2\varphi\right)}}}{\frac{1}{\delta e}} \right\}_{\delta e = 0} = \left. \frac{\int_0^{\lrcorner} -\frac{8r^2 \delta e \sin^2\varphi\, d\varphi}{(4r^2 + \delta e^2)^2 \lrcorner\varphi^3}}{-\frac{1}{\delta e^2}} \right\}_{\delta e = 0} = 0.$$

ON THE ERRORS INCIDENT TO VALUES INTERPOLATED BY MEANS OF FIRST DIFFERENCES FROM TABLES OF LOGARITHMS, NATURAL TRIGONOMETRIC FUNCTIONS, ETC.

By R. S. Woodward, Washington, D. C.

The theory of the errors incident to values interpolated by means of first differences from a table of logarithms, natural sines, etc., appears to be a matter of considerable difficulty. Our text-books are notably deficient in information on this subject or notably erroneous in such information as they give; indeed few writings on the subject are free from error.

The most common departure from correct procedure in treating this class of errors is in assuming the applicability to them of the law of error of least squares. In addition, the magnitude and continuity or discontinuity of the errors involved are frequently misunderstood. Here also as in other branches of the theory of errors, the actual, probable, mean, and average errors are apt to be confounded with one another.

Since attention has recently been called in the Annals by Prof. Howe to the errors incident to the most frequently employed species of interpolated values, it may be of interest to point out some of the essential features of the correct theory of such errors.

In a paper by the writer published in *The Analyst*, 1882, four species of interpolated values were considered. Two of these depended on the tabular differences and two on the true differences. It was shown that the latter are less precise than the former, or, paradoxical as it may seem, that the use of the true differences is less advantageous than the use of the tabular differences. Of the other two species, the first retains one or more figures beyond the tabular number of figures, and the second (which is most commonly employed) rounds the interpolated value to the nearest unit of the last tabular place. It will be convenient to refer to these two as the the first and second species.

The theory of the errors of the first species has been fully worked out by Bessel (*Astronomische Nachrichten*, Nos. 358 and 359, 1838) and affords a basis for the theory of the errors of the second species.

I am not aware that the theory of the errors of the second class has been completely developed. In the paper referred to above, however, I have indicated the process of deriving the laws of frequency of the errors of the second species and have given the laws and consequent probable errors for several special cases. I am now able to extend the process and give more general results.

The actual error ε_0 of an interpolated value of the first species is expressed by the equation

$$\varepsilon_0 = (1 - t)\,\varepsilon_1 + t\,\varepsilon_2, \qquad (1)$$

in which ε_1 and ε_2 are the actual errors of the tabular logarithms or numbers, and t is the interpolating factor, which varies from 0 to 1. ε_1 and ε_2 are expressed in units of the last tabular place and may have any value from $-\frac{1}{2}$ to $+\frac{1}{2}$; hence also ε_0 may have any value between the limits $-\frac{1}{2}$ and $+\frac{1}{2}$.

The actual error ε of an interpolated value of the second species will equal that for the first species plus the error which comes from rounding the product of the interpolating factor and the tabular difference to the nearest unit. Hence, if we call the error from the latter source ε_3, we shall have

$$\varepsilon = (1 - t)\,\varepsilon_1 + t\,\varepsilon_2 + \varepsilon_3. \qquad (2)$$

It must now be observed that ε_3 differs from ε_1 and ε_2 in two important respects, viz: 1°, it is discontinuous, and 2°, it is dependent on t. Thus for $t = \frac{1}{2}$, ε_3 must be either 0 or $\frac{1}{2}$. Again, for $t = \frac{1}{3}$, ε_3 is restricted to one of three values, viz: 0, $+\frac{1}{3}$ and $-\frac{1}{3}$. On this account the maximum value of ε will vary also with t. For $t = \frac{1}{2}$ the maximum value of $\varepsilon = \frac{1}{2} + \frac{1}{2} = 1$, but for $t = \frac{1}{3}$ the maximum of $\varepsilon = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$; this maximum cannot always be 1 as some of the books assert. Moreover, if t is small relatively to the largest tabular difference for the table used, ε_3 will be correspondingly small. Thus, in a 5-place table of logarithms, the largest tabular difference is 44; and if $t = 0.001$ the maximum value of ε_3 is 0.044 and the maximum of ε is 0.544.*

To determine the probable, mean, and average errors corresponding to ε_0 and ε_1 we must know the laws of frequency of these actual errors. If we represent the probability of the occurrence of an error ε by $\varphi(\varepsilon)\,d\varepsilon$, it may be shown that† for the system of errors represented by equation (1)

$$\begin{aligned}
\varphi(\varepsilon_0) &= \frac{\frac{1}{2} + \varepsilon_0}{(1 - t)\,t} \text{ for values of } \varepsilon_0 \text{ between } -\tfrac{1}{2} \text{ and } -(\tfrac{1}{2} - t), \\
&= \frac{1}{1 - t} \text{ for values of } \varepsilon_0 \text{ between } -(\tfrac{1}{2} - t) \text{ and } +(\tfrac{1}{2} - t), \qquad (3) \\
&= \frac{\frac{1}{2} - \varepsilon_0}{(1 - t)\,t} \text{ for values of } \varepsilon_0 \text{ between } +(\tfrac{1}{2} - t) \text{ and } +\tfrac{1}{2}.
\end{aligned}$$

As our present purpose is to discuss the errors of the second species, we shall not derive the probable and other critical errors of the first species. It is sufficient to remark that these critical errors show that an interpolated value of the first species is in general much more precise than a tabular value.†

To derive the laws of frequency of the errors ε of the second species, it is necessary to combine the continuous errors ε_0, whose laws of frequency are

*These facts would seem to be perfectly plain, but experience justifies the assertion that they are not commonly appreciated.

†See Bessel's paper or that of the writer referred to above.

given by (3), with the discontinuous errors ε_3. First consider the case in which t is a fraction of the form

$$t = \frac{1}{n},$$

in which n is an integer.* Then the possible values of ε_3 are the following : — .

$$\left.\begin{aligned} &0, +\frac{1}{n}, +\frac{2}{n}, \ldots\ldots +\frac{n-1}{2n} \\ &-\frac{1}{n}, -\frac{2}{n}, \ldots\ldots -\frac{n-1}{2n} \end{aligned}\right\} \text{ for } n \text{ odd,}$$

$$\left.\begin{aligned} &0, +\frac{1}{n}, +\frac{2}{n}, \ldots\ldots +\frac{n-2}{2n} \\ &-\frac{1}{n}, -\frac{2}{n}, \ldots\ldots -\frac{n-2}{2n} \\ &\pm\frac{1}{2} \end{aligned}\right\} \text{ for } n \text{ even.}$$

These errors are all equally likely to occur. The sign of the error $\frac{1}{2}$ for n even is arbitrary. It will here be considered positive only; and, in deriving the laws of frequency of the errors ε for n even, they will be considered with respect to magnitude simply, or as if they all had the same sign. It will be observed that the numerical maximum of ε is

$$\pm\left(\frac{1}{2} + \frac{n-1}{2n}\right) = \pm\frac{2n-1}{2n} \text{ for } n \text{ odd, and}$$

$$+\tfrac{1}{2} + \tfrac{1}{2} = +1 \text{ for } n \text{ even.}$$

Now if we add each of the above n values of ε_3 to ε_0 in (3), we shall have n sets of probability functions, $\varphi(\varepsilon_0 + \varepsilon_3) = \varphi(\varepsilon)$. If then the element-areas $\varphi(\varepsilon)\,d\varepsilon$ defined by these several functions be summed for every value of ε, the result will be the probability area for the whole system of errors. The area enclosed by the straight lines whose equations are given by (3) and the axis of ε_0, is 1. Hence the sum of n such unit areas is n, and we must divide the results of the above summation by n in order to have the integral of $\varphi(\varepsilon)\,d\varepsilon$ over the whole range of error, or the sum of the probabilities of all possible errors ε, unity. Without going further into the details of this process, which may be easily illustrated and worked out graphically, the results are as follows : —

For n odd

$$\varphi(\varepsilon) = \frac{n}{n-1}\left(\frac{2n-1}{2n} + \varepsilon\right) \text{ for values of } \varepsilon \text{ between } -\frac{2n-1}{2n} \text{ and } -\frac{1}{2n}, \quad (4)$$

*For a given table n must not exceed twice the greatest tabular difference.

$$= 1 \qquad \text{for values of } \varepsilon \text{ between } -\frac{1}{2n} \text{ and } +\frac{1}{2n}, \quad (4\ \textit{cont.})$$

$$= \frac{n}{n-1}\left(\frac{2n-1}{2n} - \varepsilon\right) \qquad \text{for values of } \varepsilon \text{ between } +\frac{1}{2n} \text{ and } +\frac{2n-1}{2n}.$$

For n even

$$\varphi(\varepsilon) = \frac{n}{n-1}\left(\frac{2(n-1)}{n} - \varepsilon\right) \qquad \text{for values of } \varepsilon \text{ between } 0 \text{ and } \frac{1}{n},$$

$$= \frac{2n}{n-1}\left(\frac{2n-1}{2n} - \varepsilon\right) \qquad \text{for values of } \varepsilon \text{ between } \frac{1}{n} \text{ and } \frac{n-1}{n}, \qquad (5)$$

$$= \frac{n}{n-1}(1-\varepsilon) \qquad \text{for values of } \varepsilon \text{ between } \frac{n-1}{n} \text{ and } 1.$$

For n odd the probability area is a trapezoid symmetrical with respect to the axis of $\varphi(\varepsilon)$. For n even the area is bounded by three straight lines and the positive axes of ε and $\varphi(\varepsilon)$, the errors as stated above being considered with respect to magnitude only.

Now let ε_p = the probable error,

ε_m = the mean error,

ε_a = the average error.

The probable error is derived from the condition

$$\int_0^{\pm\varepsilon_p} \varphi(\varepsilon)\, d\varepsilon = \tfrac{1}{4} \text{ for } n \text{ odd},$$

$$= \tfrac{1}{2} \text{ for } n \text{ even}.$$

The square of the mean error

$$\varepsilon_m^2 = 2\int_0^{\frac{2n-1}{2n}} \varphi(\varepsilon)\, \varepsilon^2\, d\varepsilon \quad \text{for } n \text{ odd},$$

$$= \int_0^{1} \varphi(\varepsilon)\, \varepsilon^2 d\varepsilon \qquad \text{for } n \text{ even}.$$

The average error

$$E_a = 2\int_0^{\frac{2n-1}{2n}} \varphi(\varepsilon)\, \varepsilon d\varepsilon \quad \text{for } n \text{ odd},$$

$$= \int_0^1 \varphi(\varepsilon)\, \varepsilon d\varepsilon \quad \text{for } n \text{ even.}$$

In applying these formulæ, attention must be paid to the several values of $\varphi'(\varepsilon)$ and the corresponding limits falling between those given in the above integrals. They give

$$\left.\begin{aligned} \varepsilon_p &= \frac{2n+1}{2n} - \frac{1}{2n}\sqrt{[2n(n-1)]} \\ \varepsilon_m &= \sqrt{\left(\frac{(2n-1)^4 - 1}{96n^3(n-1)}\right)} \\ \varepsilon_a &= \frac{(2n-1)^3 - 1}{24n^2(n-1)} \end{aligned}\right\} \; n \text{ odd}; \qquad (6)$$

$$\left.\begin{aligned} \varepsilon_p &= 1 - \tfrac{1}{2}\sqrt{2} \quad \text{for } n = 2 \\ \varepsilon_p &= \frac{2n-1}{2n} - \frac{1}{2n}\sqrt{[2n(n-1)-1]} \\ &\quad \text{for } n > 2 \\ \varepsilon_m &= \sqrt{\left(\frac{n^4 + (n-1)^4 - 1}{12n^3(n-1)}\right)} \\ \varepsilon_a &= \frac{n^3 + (n-1)^2 - 1}{6n^2(n-1)} \end{aligned}\right\} \; n \text{ even.} \qquad (7)$$

For obvious reasons these formulæ, for n odd and even, should give identical results for ε_p, ε_m, and ε_a, when $n = \infty$; moreover these results should be the same as those for $n = 2$. Such is the case. The value $n = \infty$ is not permissible, however, in these formulæ when applied to such tables as we use, since the tabular differences are always finite. To give consistent results when applied to such tables, n must be restricted to values not exceeding twice the greatest tabular difference. For values of n greater than this limit, the probable, mean, and average errors will be less than those given by (6) and (7) and will approach those for the system expressed by equation (1) as n approaches infinity or unity. The process outlined above may be readily applied to determine the laws of frequency of ε when n exeeeds the limit just mentioned.

As to the more general case, in which the interpolating factor t is a proper fraction of the form m/n, m and n being integers, a moment's consideration will show that it is comprised practically, or so far as numerical values are concerned, in the first case where $t = 1/n$. For, supposing m and n to have no common factor, the possible values of ε_3 corresponding to m/n are identical with those for $1/n$. Thus for $m/n = \frac{3}{7}$ the values of ε_3 are the same as those for $m/n = \frac{1}{7}$; those for $\frac{2}{10}$ and $\frac{4}{10}$ the same as those for $\frac{1}{5}$; those for $\frac{3}{10}$ the same as those for

$\frac{1}{10}$, and so on. The values of $\varphi(\varepsilon_0)$ and hence $\varphi(\varepsilon)$ for $t = m/n$ will indeed differ from those when $t = 1/n$. $\varphi(\varepsilon)$ will be of a somewhat more complex character.* But the limiting values of ε are the same in the two cases, and hence the probability functions cannot give probable, mean, or average errors differing materially in numerical value from those when $t = 1/n$. The restriction mentioned above relative to the magnitude of n for a given table must be borne in mind.

The following table gives the probable, mean, average, and maximum actual errors corresponding to the interpolating factor $t = 1/n$ for values of n from 1 to 10. The values of these errors for $n = 1$, or for a tabular value, are given in the first line for the sake of ready comparison.

Interpolating Factor.	Probable Error.	Mean Error.	Average Error.	Maximum Actual Error.
1	0.250	0.289	0.250	$\frac{1}{2}$
$\frac{1}{2}$	0.292	0.408	0.333	1
$\frac{1}{3}$	0.256	0.347	0.287	$\frac{5}{6}$
$\frac{1}{4}$	0.276	0.382	0.313	1
$\frac{1}{5}$	0.268	0.370	0.303	$\frac{9}{10}$
$\frac{1}{6}$	0.277	0.385	0.315	1
$\frac{1}{7}$	0.274	0.380	0.311	$\frac{13}{14}$
$\frac{1}{8}$	0.279	0.389	0.318	1
$\frac{1}{9}$	0.278	0.386	0.316	$\frac{17}{18}$
$\frac{1}{10}$	0.281	0.392	0.320	1

The mean of the average errors in the above table for $t = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}$, and $\frac{5}{10}$, using $\frac{1}{10}$ and $\frac{1}{5}$ as arguments in the place of $\frac{3}{10}$ and $\frac{4}{10}$, is 0.316. This number coincides closely, as it should, with the average of the 1000 actual errors given by Prof. Howe in No. 6 of the ANNALS.

*Thus, for example, if $t = \frac{2}{5}$ the probability function is

$\varphi(\varepsilon) = 1$ for values of ε between 0 and $\pm \frac{1}{10}$,
$= \frac{5}{6}(\frac{13}{10} \pm \varepsilon)$ for values of ε between $\pm \frac{1}{10}$ and $\pm \frac{3}{10}$,
$= \frac{5}{3}(\frac{1}{5} \pm \varepsilon)$ for values of ε between $\pm \frac{3}{10}$ and $\pm \frac{7}{10}$,
$= \frac{5}{6}(\frac{9}{10} \pm \varepsilon)$ for values of ε between $\pm \frac{7}{10}$ and $\pm \frac{9}{10}$;

and the probable, mean, and average errors are 0.261, 0.351, and 0.290 respectively.

ON THE COMPUTATION OF THE POSITION IN ORBIT OF A PARTICLE IN THE TAIL OF A COMET.

By Prof. Ormond Stone, University of Virginia.

We have the well known equations

$$f_1^2 = k^2 \left(\frac{2}{r_1} - \frac{1}{a} \right), \tag{1}$$

$$f_1 \cos \beta_1 = \frac{k}{\sqrt{p}} e \sin v_1, \tag{2}$$

$$f_1 \sin \beta_1 = \frac{k \sqrt{p}}{r_1}; \tag{3}$$

where, neglecting the mass of the comet,

k is the constant of the solar system,

e, a, p, the eccentricity, semi major axis, and semi parameter of the orbit of the comet,

r_1, v_1, the radius vector and true anomaly for any given time t_1,

f_1, the corresponding velocity in orbit, and

β_1, the angle which the radius vector makes with the tangent.

(2) and (3) give $\tan \beta_1 = \frac{p}{r_1} \cdot \frac{1}{e \sin v_1} = \frac{1 + e \cos v_1}{e \sin v_1}$. (a)

For a free particle moving with the comet up to the time t_1, but after that moving in a hyperbolic orbit subject to a repulsive force $k^2 \varkappa$ acting in addition to gravity, but in opposition thereto, we have

$$f^2 = k^2 (1 - \varkappa) \left(\frac{2}{R} + \frac{1}{b} \right), \tag{4}$$

$$f \cos \beta = \frac{k \sqrt{(1 - \varkappa)}}{\sqrt{P}} E \sin V, \tag{5}$$

$$f \sin \beta = \frac{k \sqrt{(1 - \varkappa)} \sqrt{P}}{R}; \tag{6}$$

in which b is the hyperbolic semi-major axis.

(1) and (4) give $\frac{2}{r_1} - \frac{1}{a} = (1 - \varkappa) \left(\frac{2}{r_1} + \frac{1}{b} \right)$,

since at the time t_1, $f = f_1$ and $R = r_1$;

$$\therefore b = \frac{1 - \varkappa}{2\varkappa} \cdot \frac{r_1}{1 - \frac{r_1}{2a\varkappa}}, \qquad (b)$$

$$= \frac{1 - \varkappa}{2\varkappa} r_1, \text{ when the orbit of the comet is a parabola.}$$

Putting $E = \sec \psi$, whence $P = b \tan^2 \psi$, (1) and (6) give

$$f_1 \sin {}_1\beta_1 = k\sqrt{\left(\frac{2}{r_1} - \frac{1}{a}\right)} \cdot \sin {}_1\beta_1 = k_1 \sqrt{(1 - \varkappa)} \sqrt{\frac{b}{r_1}} \tan \psi;$$

whence, substituting the value of b given in (b) and solving, we obtain

$$\tan \psi = \frac{2\sqrt{\varkappa}}{1 - \varkappa} \cdot \sqrt{\left(1 - \frac{r_1}{2a}\right)} \cdot \sqrt{\left(1 - \frac{r_1}{2a\varkappa}\right)} \cdot \sin {}_1\beta_1, \qquad (c)$$

$$= \frac{2\sqrt{\varkappa}}{1 - \varkappa} \sin {}_1\beta_1, \text{ when the orbit of the comet is a parabola.}$$

The sign of E is assumed the same as that of $\tan \psi$, hence negative when $\varkappa > 1$.

(5) and (6) give $$\tan {}_1\beta = \frac{P}{E} \cdot \frac{1}{R \sin V}, \qquad (7)$$

where, if F be the hyperbolic anomaly,

$$R \sin V = b \tan \psi \tan F.$$

Substituting in (7) the values of E, P, and $R \sin V$, and simplifying, we have

$$\tan F = \sin \psi \cot {}_1\beta. \qquad (d)$$

or, for the time t_1, $$\tan F_1 = \sin \psi \cot {}_1\beta_1. \qquad (d_1)$$

After computing the formulæ (a), (b), (c), and (d_1), r_1 and V_1 are found from the well known equations

$$r_1 \sin V_1 = b \tan \psi \tan F_1,$$
$$r_1 \cos V_1 = b (E - \sec F_1).$$

The agreement of this value of r_1 with that derived directly from the orbit of the comet will form a convenient check.

The mean motion and time of hyperbolic perihelion passage t_0 is found from the equations

$$n = \frac{\lambda k_1 \sqrt{(1 - \varkappa)}}{b^{\frac{3}{2}}},$$

$$N_1 = n(t_1 - t_0) = \lambda E \tan F_1 - \log \tan (45^\circ + \tfrac{1}{2}F_1),$$

where λ is the modulus of common logarithms.

The position in orbit for any given time t is found from the formulæ

$$\left.\begin{aligned} \lambda E \tan F - \log \tan (45^\circ + \tfrac{1}{2}F) &= n(t - t_0) = N, \\ R \sin V &= b \tan \psi \tan F, \\ R \cos V &= b(E - \sec F). \end{aligned}\right\} \qquad (e)$$

F is found from the first of these equations by successive approximation. If $E \sec F$ be negative or be positive and large, $1/(E \sec F - 1)$ will be small, and we may obtain $\tan F_1$ from the equation

$$\tan F_1 = \frac{N + \log \tan (45^\circ + \tfrac{1}{2}F_0)}{\lambda E},$$

where F_0 is an assumed value of F. Subtracting this from

$$\tan F = \frac{N + \log \tan (45^\circ + \tfrac{1}{2}F)}{\lambda E},$$

we have, including only terms of the first order,

$$F = F_1 + \frac{F_1 - F_0}{E \sec F - 1}.$$

If F be not too large we may express the first of equations (e) in the form

$$E \cdot \tfrac{1}{2}(e^x - e^{-x}) - x = n_0(t - t_0) = N_0,$$

where $$\lambda x = \log \tan (45^\circ + \tfrac{1}{2}F)$$

and $$n_0 = \frac{k_1 (1 - \varkappa)}{b^{\frac{3}{2}}}.$$

Since $$\frac{e^x - e^{-x}}{2} = \frac{x}{1} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots,$$

if we put $$\mu^2 = 1 + \frac{x^2}{4 \cdot 5} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} + \ldots,$$

we have after reduction

$$x^3 + \frac{6(E-1)}{E} \cdot \frac{1}{\mu^2} \cdot x - \frac{6N_0}{E} \cdot \frac{1}{\mu^2} = 0,$$

a cubic equation, which may be solved by means of the following formulæ:—

$$a_0 = 2(E-1)/E, \quad a_1 = E/3N_0;$$
$$a = a_0 a_1^{\frac{3}{2}}, \qquad \beta = 2a_0^{\frac{1}{2}};$$
$$\tan \theta = a/\mu, \quad \tan \varphi = \tan^{\frac{1}{3}} \tfrac{1}{2}\theta, \quad x = \beta/\mu \tan \varphi.$$

μ is obtained from the accompanying table with the argument x^2. In the first approximation, unless otherwise known, μ may be assumed equal to unity.

x^2	log μ	Diff.	x^2	log μ	Diff.	x^2	log μ	Diff.	x^2	log μ	Diff.
0.0	0.00000		1.0	0.01084		2.0	0.02166		3.0	0.03245	
		109			109			108			108
0.1	0.00109		1.1	0.01193		2.1	0.02274		3.1	0.03353	
		108			108			108			107
0.2	0.00217		1.2	0.01301		2.2	0.02382		3.2	0.03460	
		109			108			108			108
0.3	0.00326		1.3	0.01409		2.3	0.02490		3.3	0.03568	
		108			108			108			108
0.4	0.00434		1.4	0.01517		2.4	0.02598		3.4	0.03676	
		109			109			108			107
0.5	0.00543		1.5	0.01626		2.5	0.02706		3.5	0.03783	
		108			108			108			108
0.6	0.00651		1.6	0.01734		2.6	0.02814		3.6	0.03891	
		108			108			108			107
0.7	0.00759		1.7	0.01842		2.7	0.02922		3.7	0.03998	
		109			108			107			108
0.8	0.00868		1.8	0.01950		2.8	0.03029		3.8	0.04106	
		108			108			108			107
0.9	0.00976		1.9	0.02058		2.9	0.03137		3.9	0.04213	
		108			108			108			108
1.0	0.01084		2.0	0.02166		3.0	0.03245		4.0	0.04321	

SOLUTIONS OF EXERCISES.

22

THE dome of the rotunda of the University of Virginia is spherical. The length of its meridian section is $85'.2$ and the girth of its base is $214'$. From these data it is required to compute the radius and the surface of the dome.

[*William M. Thornton.*]

SOLUTION.

We have for the chord of the meridian arc

$$\frac{214}{\pi} = 68.1.$$

If therefore x be the half central angle of the arc,

$$\frac{\sin x}{x} = \frac{681}{852} = 0.799.$$

Passing to logarithms and expressing x in minutes by putting

$$x = \mathrm{x} \cdot \frac{\pi}{10800},$$

we get $$y = \log \sin x - \log \mathrm{x} + 3.6336 = 0.$$

By trial we find, if

$$x = 64^\circ, \quad y = +0.0030;$$
$$x = 65^\circ, \quad y = -0.0002;$$

therefore $x = 64°56'$ approximately and

$$R = \frac{68.1}{2 \sin x} = 37.6.$$

For the height of the dome we have

$$H = R(1 - \cos x) = 21.6,$$

and for the surface

$$S = 2\pi RH = 5100^{sq}.$$

[*W. M. Thornton.*]

30

INVESTIGATE formulæ for the logarithms of 13, 19, and 73 in terms of the logarithms of primes less than 13 and of $(N + 1)/N$ where $N = 132495$; 262143; 274625. [*F. H. Loud.*]

SOLUTION.

Put $x = \log \frac{132496}{132495}$, $y = \log \frac{262144}{262143}$, $z = \log \frac{274626}{274625}$;

then $3 \log 13 = 19 \log 2 + \log 11 - \log 3 - 3 \log 5 - \log 7 - y - z$,

$\log 73 = 4 \log 2 - \log 3 - \log 5 + 2 \log 7 - 2 \log 11 + 2 \log 13 - x$,

$\log 19 = 3 \log 5 + 3 \log 13 - \log 2 - 2 \log 3 - \log 11 - \log 73 + z$.

[*F. H. Loud.*]

38

SHOW that among any three roots a, b, c of a biquadratic, as $x^4 + px^2 + b = 0$, this relation holds: —

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c}.$$

SOLUTION.

Since the terms containing x and x^3 are wanting in the biquadratic given, we have

$$abc + abd + acd + bcd = 0,$$

$$a + b + c + d = 0,$$

where d is the remaining root;

$$\therefore -\frac{1}{d} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a + b + c}.$$

[*Asher B. Evans.*]

48

In the angle A a point P is given. Construct a triangle ABC whose base BC shall contain P, while the sum of the sides AB and AC equals a given length. [*W. M. Thornton.*]

SOLUTION.

On AB lay off $AD =$ the given length. The problem requires to draw a line BC through P so as to make $BD = AC$. This can readily be effected by the tentative process known as the *méthode de fausse position*, in the following manner: Through P draw any three lines cutting AD in B_1, B_2, B_3; on AC lay off $AC_1 = B_1D$, $AC_2 = B_2D$, $AC_3 = B_3D$. The two concentric pencils, $P(B_1, B_2, B_3)$ and $P(C_1, C_2, C_3)$, being in perspective position with the two identical rows B_1, B_2, B_3 and C_1, C_2, C_3 respectively, are projective. Their two common rays, if real, may easily be constructed by means of an auxiliary circle, in the usual way. Each one of these common rays satisfies the conditions of the problem. The construction may, of course, be simplified by selecting particular lines for the three arbitrary rays; e. g. by taking PD and the two parallels to AB and AC for PB_1, PB_2, PB_3 respectively. If, in addition, the arbitrary circle through P be made to pass through A and D, it will be easy to determine the conditions of the possibility of a real solution. [*Alexander Ziwet.*]

50

ABC is an equilateral triangle inscribed in a circle. From any point P within the angle BAC, straight lines PA, PB, PC are drawn. Show, *geometrically*, that $PB + PC - PA$ is a minimum when P is on the circumference of the circle about ABC. [*R. D. Bohannan.*]

SOLUTION.

Let AP cut the circumference in P'. Then [*Eu.* 6*D*]

$$P'B + P'C = P'A.$$

Draw BP', CP' and drop on them perpendiculars PX, PY. Then, since the angles $PP'X$, $PP'Y$ are each 60°, $P'X$ and $P'Y$ each equal half PP', whence

$$P'X + P'Y = P'P;$$
$$\therefore BX + CY = PA.$$

But BX, CY are less than BP, CP except when P is on the circumference; hence $BP + CP - AP$ is greater than zero except when P is on the circumference. [*R. D. Bohannan.*]

51

A QUADRILATERAL, both circumscriptible and inscriptible, has a given perimeter $4s$ and one given angle a. Show that its area is

$$2s^2 : (\operatorname{cosec} a + \operatorname{cosec} x),$$

where x is one of the angles adjacent to a. And find x when the area is greatest.

[*W. M. Thornton.*]

SOLUTION I.

Let r = radius of inscribed circle.

The sum of two opposite sides $= r\left(\cot\frac{x}{2}+\cot\frac{a}{2}+\tan\frac{x}{2}+\tan\frac{a}{2}\right)=2s$;

$$\therefore \text{Area} = 2rs = \frac{4s^2}{\cot\frac{a}{2}+\tan\frac{a}{2}+\cot\frac{x}{2}+\tan\frac{x}{2}} = 2s^2:(\operatorname{cosec} a+\operatorname{cosec} x).$$

The area is greatest when $\operatorname{cosec} x$ has its least positive value; i. e. when $x=\frac{\pi}{2}$.

[*R. H. Graves.*]

SOLUTION II.

Let $l, a, m, x, n, \pi-a, p, \pi-x$ be in order the sides and angles of the quadrilateral. Since it is circumscriptible,

$$l+n=m+p=2s,$$

and we have for the area the equivalent expressions

$$A=\frac{lm+np}{2\operatorname{cosec} a}=\frac{mn+lp}{2\operatorname{cosec} x}=\frac{lm+mn+np+lp}{2(\operatorname{cosec} a+\operatorname{cosec} x)}=\frac{(l+n)(m+p)}{2(\operatorname{cosec} a+\operatorname{cosec} x)};$$

$$\therefore A=\frac{2s^2}{\operatorname{cosec} a+\operatorname{cosec} x}.$$

This expression is greatest when $\operatorname{cosec} x$ is least; i. e. when $x=90°$.

[*R. D. Bohannan.*]

52

FIND the height to which the Washington Monument must be built so that a body placed on top of it would have no weight. [*A. Hall.*]

SOLUTION I.

Let x be the distance from the centre of the earth to the top of the monument,

R be the radius of the earth,
M mass of the earth,
T time of rotation of the earth,
φ latitude of the monument,
m mass of the moon,
a mean distance of the moon,
t periodic time of the moon.

Since the attractive force of the earth must balance the vertical component of the centrifugal force on a body at the top of the monument,

$$\frac{M}{x^2} = \frac{4\pi^2}{T^2} x \cos^2 \varphi ;$$

$$\therefore x^3 = M \cdot \frac{T^2}{4\pi^2} \cdot \sec^2 \varphi .$$

But $$a^3 = (M + m) \frac{t^2}{4\pi^2};$$

$$\therefore \left(\frac{x}{a}\right)^3 = \frac{M}{M+m} \cdot \left(\frac{T}{t}\right)^2 \sec^2 \varphi .$$

Putting $M = 81m$, $T = 1$, $t = 27.39647$, $\varphi = 38°53'18''$, $a = 238800^m$, we get

$$x = 30927^m,$$

and the height required is $30927^m - 3958^m = 26969^m$. [*A. Hall.*]

SOLUTION II.

With the same notation we have for the weight of unit mass at height x above the centre of the earth

$$W \cdot \frac{R^2}{x^2} - \frac{4\pi^2}{T^2} x \cos^2 \varphi ,$$

where W is its weight on a still earth at the surface. Hence at the surface

$$g = W - \frac{4\pi^2}{T^2} R \cos^2 \varphi ,$$

and at the top of the monument

$$\left(\frac{x}{R}\right)^3 = \frac{T^2}{4\pi^2} \cdot \frac{W}{R} \sec^2 \varphi .$$

Eliminating W we get $$\left(\frac{x}{R}\right)^3 = 1 + \frac{g}{R} \cdot \frac{T^2}{4\pi^2} \cdot \sec^2 \varphi .$$

Putting $T = 86400^s$, $g = 32.15^{f/s}$, $R = 20889000^f$, $\varphi = 38°53'18''$;

we get $$x = 7.837R,$$

and the height required is $6.837R = 27050^m$. [*Chas. Puryear.*]

[A solution nearly the same, but not quite accurate in method, was sent by S. T. Moreland.—*Ed.*]

53

SHOW that when $x = \frac{2\pi}{13}$

$$(\cos x + \cos 5x)(\cos 2x + \cos 3x)(\cos 4x + \cos 6x) = -\tfrac{1}{8}.$$

[*R. H. Graves.*]

SOLUTION I.

Converting the sums of cosines into products, the equation to be proved is

$$\cos 3x \cos 2x \cos 2\tfrac{1}{2}x \cos \tfrac{1}{2}x \cos 5x \cos x = -\tfrac{1}{64},$$

or denoting the product by X and putting $\omega = \frac{1}{2}x = \frac{2\pi}{26}$,

$$X = \cos\omega \cos 2\omega \cos 4\omega \cos 5\omega \cos 6\omega \cos 10\omega = -\tfrac{1}{64}.$$

Noticing that 3ω, 7ω, 8ω, 9ω, 11ω, and 12ω are the supplements of the six angles which occur in X, we see that

$$X^2 = \cos\omega \cos 2\omega \cos 3\omega \ . \ . \ . \ . \cos 12\omega;$$

and since 14ω, 15ω, . . . 23ω have the same cosines as the angles which occur in X^2 we have, since $\cos 0 = 1$ and $\cos 13\omega = -1$,

$$-X^4 = \cos 0 \cos\omega \cos 2\omega \ . \ . \ . \ . \cos 23\omega.$$

I have given in the *Analyst*, Vol. VI, p. 105, the equation whose roots are the sines of the n angles $\alpha_0, \alpha_1, \ . \ . \ . \ \alpha_{n-1}$ included in the expression $\alpha_0 + k\frac{2\pi}{n}$ $[n = 0, 1, \ . \ . \ . \ n-1]$. From this equation it appears that the product of these sines is, when n is even,

$$(-1)^{\frac{n}{2}} \sin^2 \tfrac{1}{2}n\alpha.$$

Putting $\alpha = \frac{\pi}{2}$ and $n = 26$, this product becomes the value of $-X^4$ above; thus

$$-X^4 = -\frac{1}{2^{24}}; \quad \text{whence } X = \pm\frac{1}{2^6};$$

or since X contains but one negative factor, $X = -\frac{1}{64}$, which was to be proved.

[The equation referred to above is that which expresses that $\cos n\alpha$ has the same value for each of the angles in question, for when n is even the series for $\cos n\alpha$ in powers of $\sin\alpha$ terminates with $\sin^n\alpha$. When n is odd the corresponding equation is derived from the value of $\sin n\alpha$. The article in the *Analyst* contains a discussion of the symmetrical functions of the sines of the angles in question.] [*Wm. Woolsey Johnson.*]

SOLUTION II.

In Burnside and Panton's *Theory of Equations*, p. 99, the example occurs: "Form the cubic whose roots are $\alpha + \alpha^8 + \alpha^{12} + \alpha^5$, $\alpha^2 + \alpha^3 + \alpha^{11} + \alpha^{10}$, $\alpha^4 + \alpha^6 + \alpha^9 + \alpha^7$, where α is an imaginary root of $x^{13} - 1 = 0$."

The required equation is found by the usual process to be

$$x^3 + x^2 - 4x + 1 = 0. \qquad (1)$$

The given roots may be written

$$\alpha + \alpha^{-1} + \alpha^5 + \alpha^{-5}, \quad \alpha^2 + \alpha^{-2} + \alpha^3 + \alpha^{-3}, \quad \alpha^4 + \alpha^{-4} + \alpha^6 + \alpha^{-6},$$

or $\quad 2(\cos\theta + \cos 5\theta), \quad 2(\cos 2\theta + \cos 3\theta), \quad 2(\cos 4\theta + \cos 6\theta),$

where $$\theta = \frac{2\pi}{13};$$

therefore by (1)

$$8(\cos\theta + \cos 5\theta)(\cos 2\theta + \cos 3\theta)(\cos 4\theta + \cos 6\theta) = -1.$$

Other relations may evidently be deduced. [*R. H. Graves.*]

SOLUTION III.

Multiplying together the factors of the left hand member, reducing each product of cosines to sums of cosines, and substituting $13 - n$ for n (any coefficient of x) whenever n is greater than 6, we have

$$\tfrac{1}{2} + \tfrac{5}{4}\sum_{n=1}^{n=6}\cos nx = \tfrac{1}{2} + \tfrac{5}{4}\frac{\sin 3x \cos 3\tfrac{1}{2}x}{\sin \tfrac{1}{2}x} \quad \text{(See Snowball's } \textit{Trigonometry}\text{, p. 84)}$$

$$= \tfrac{1}{2} - \tfrac{5}{4}\frac{\sin 3x \cos 3x}{\sin \tfrac{1}{2}x} = \tfrac{1}{2} - \tfrac{5}{8}\frac{\sin 6x}{\sin \tfrac{1}{2}x} = -\tfrac{1}{8},$$

since $\pi = 6\frac{1}{2}x$. [*Ormond Stone.*]

54

SHOW that if in a plane triangle ABC

$$\cos A + \cos B + \cos C = 1/2,$$

the centre of the circumscribed circle lies on the circumference of the inscribed circle. [*R. D. Bohannan.*]

SOLUTION.

Let $R =$ radius of circumscribed circle, and $r =$ radius of inscribed circle. Then (Chauvenet's *Trigonometry*, pp. 78 and 79)

$$\frac{r}{R} = \cos A + \cos B + \cos C - 1. \qquad (1)$$

$$D^2 = R^2 - 2Rr,$$

D being the distance between the centres of the two circles. By substituting the value of R found from (1) in this last equation, since $\cos A + \cos B + \cos C = 1/2$, there results after reduction $D = r$; hence the proposition is proved.

[*P. Richardson.*]

55

WHAT is the locus of the vertex of a variable angle whose sides intercept a fixed line and whose bisectrix passes through a given point in that line.

[*L. G. Carpenter.*]

SOLUTION I.

Let a, na be the segments of the fixed line,

x, y co-ordinates of the vertex of the moving angle.

Let the origin be taken at the point of division of the fixed line which is the x-axis. The sides bear to each other the ratio $1 : n$. Hence

$$(x - na)^2 + y^2 = n^2[(x + a)^2 + y^2];$$

$$\therefore x^2 + y^2 = \frac{2n}{1 - n} ax.$$

The locus therefore is a circle described on the segment of the fixed line obtained by dividing it harmonically in the ratio $1 : n$. [*S. J. Cunningham.*]

SOLUTION II.

The line joining the given point with its harmonic conjugate with respect to the extremities of the fixed line always subtends a right angle at the vertex of the variable angle. Therefore the locus is a circle. [*R. H. Graves.*]

56

CONSTRUCT a triangle; given its circumscribed circle and the radii which bisect its sides.

SOLUTION.

Let OD, OE, OF be the radii; produce DO to meet the circumference in D' and make FA equal to ED'; from A draw the chords AB, AC perpendicular to OF, OE, and join BC; ABC will be the required triangle.

[*R. D. Bohannan.*]

58

DIVIDE an isosceles triangle by perpendiculars from a point within it to the sides, into three equivalent quadrilaterals.

SOLUTION.

Let ABC be the given triangle, AD its altitude, P the required point which must lie in AD, and PM the perpendicular on AC; then the triangles APM, ACD are by construction similar and their areas are by hypothesis as 1 to 3; $\therefore AD = AP\sqrt{3}$.

Accordingly, to locate P we construct an equilateral triangle with AD as altitude; its base will be equal to AP. [*Charles Puryear.*]

59

SHOW that the height of a cone bears to the height H of an equal spherical dome of diameter D on the same base the ratio

$$3D - 2H : 2D - 2H.$$

SOLUTION I.

Let D denote the diameter of the sphere,
H altitude of the dome,
r radius of its base,
h altitude of the cone;

then $$r^2 = DH - H^2,$$

and the volume of the dome, being the difference of the volumes of the sector and the cone whose vertex is at the centre of the sphere, is

$$\tfrac{1}{6}\pi HD^2 - \tfrac{1}{6}\pi r^2(D - 2H) = \tfrac{1}{6}\pi H^2(3D - 2H).$$

The volume of the required cone is, again,

$$\tfrac{1}{3}\pi hr^2 = \tfrac{1}{3}\pi Hh(D - H).$$

The two volumes are equal by hypothesis;

$$\therefore \frac{h}{H} = \frac{3D - 2H}{2D - 2H}. \qquad [\textit{Charles Puryear.}]$$

SOLUTION II.

Let D_1 be the diameter of the base of the dome; the condition gives

$$\frac{\pi}{12}D_1^2 h = \frac{\pi}{8}D_1^2 H + \frac{\pi}{6}H^3;$$

$$\therefore \frac{h}{H} = \tfrac{3}{2} + \frac{2H^2}{D_1^2}.$$

But $$D_1^2 = 4H(D - H);$$

$$\therefore \frac{h}{H} = \tfrac{3}{2} + \frac{H}{2D - 2H} = \frac{3D - 2H}{2D - 2H}.$$

[*R. D. Bohannan.*]

EXERCISES.

70

A RIGHT ANGLE moves so that a given point in one side, distant c from the vertex, lies in a fixed axis, while the other side passes through a fixed point, distant c from this axis. Find the locus of the instantaneous centres or centrodes of the motion. [*R. H. Graves.*]

71

A LINE of unit length is bent to the arc of a circle such that the area of the segment it determines is a maximum. Find the radius of the circle and the form and area of the segment. [*O. Root, Jr.*]

72

CIRCLES of given radius are drawn through the focus of a fixed parabola, cutting the curve in four points Show that the products of the focal radii to these points are all equal. [*W. M. Thornton.*]

73

IF r is the radius of the inscribed circle of a triangle and ρ the radius of the triangle whose vertices are the feet of the altitudes of the given triangle,

$$r > 2\rho.$$ [*R. D Bohannan.*]

74

FOUR points are taken at random on the surface of a sphere. What is the probability, that all of the points do not lie in the same hemisphere?

[*A. Hall.*]

SELECTED.

75

REDUCE to its simplest form

$$\cot(x_0 - x_1)\cot(x_0 - x_2)\cot(x_0 - x_3) + \cot(x_1 - x_0)\cot(x_1 - x_2)\cot(x_1 - x_3) + \cot(x_2 - x_0)\cot(x_2 - x_1)\cot(x_2 - x_3).$$

76

FIND the condition that the cubic

$$6x^3 - (2n + 8)x^2 + n(n + 1)x + n(n + 1)(2 - n) = 0$$

may have equal roots.

77

CONSTRUCT a square; given one vertex and two parallel lines on which the extremities of the opposite diagonal are located.

78

A, B, C are the points of application of three parallel ~~faces~~ forces x, y, z; Q is their centre; ρ is their resultant; p, q, r are the radius vectors of A, B, C measured from an arbitrary origin M. Show that if $MQ = h$

$$h^2 = \frac{p^2x + q^2y + r^2z}{\rho} - \frac{a^2yz + b^2zx + c^2xy}{\rho^2}.$$

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